Two-loop effects in heavy flavor processes at hadron colliders

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In collaboration with:

Czakon, Sterman, Sung; Beneke, Falgari, Schwinn

Work in progress also with:

Cacciari, Mangano, Nason, Moch, Uwer

Plan of the talk

- ✓ Reporting the completion of a research program:

  - ➢ Singularities of massive gauge theory amplitudes (2 loops)
  - ➢ Soft-gluon resummation (at NNLL) in such processes
  - ➢ Collider phenomenology: top-pair production

- ✓ None of the above were known beyond one loop
- ✓ Friendly competition along the way ☺
- ✓ Besides the results we were after, unexpected questions were raised
- ✓ Everything is now settled down

A.M., Sterman, Sung '10

Singularities of Massive Gauge Theory Amplitudes

Amplitudes: the basics

- Gauge theory amplitudes: UV renormalized, S-matrix elements
- The amplitudes are not observables:
  - UV renormalized gauge amplitudes are not finite due to IR singularities.
  - Assume they are regulated dimensionally $d=4-2\varepsilon$

What was known before (massive case):

- ✓ Explicit expression for the IR poles of any one-loop amplitude derived

  Catani, Dittmaier, Trocsanyi '00

- ✓ The small mass limit is proportional to the massless amplitude

  Mitov, Moch '06
  Becher, Melnikov '07

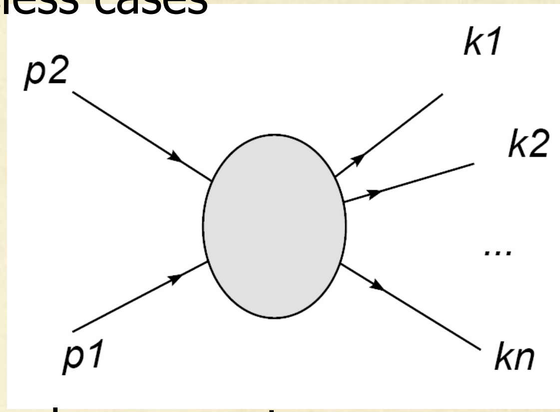
Note: predicts not just the poles but the finite parts too (for $m \to 0$)!

Factorization: “divide and conquer“

Structure of amplitudes becomes transparent thanks to factorization th.

$$M_I(\varepsilon,\mu_R,s_{ij},m_i) = J(\varepsilon,\mu_R,\mu_F,m_i)\cdot S_{IJ}(\varepsilon,\mu_R,\mu_F,s_{ij},m_i)\cdot H_J(\varepsilon,\mu_R,\mu_F,s_{ij},m_i)$$

Note: applicable to both massive and massless cases



I,J – color indexes.

J(…) – “jet” function. Absorbs all the collinear enhancement.

S(…) – “soft” function. All soft non-collinear contributions.

H(…) – “hard” function. Insensitive to IR.

Factorization: the Jet function

$$M_I(Q,m,\epsilon) = J(m,\epsilon)\cdot S_{IJ}(Q,m,\epsilon)\cdot H_J(Q,m)$$

For an amplitude with n-external legs, J(…) is of the form:

$$J(m,\epsilon) = \prod_{i=1}^{n} J_i(m,\epsilon)$$

i.e. we associate a jet factor to each external leg.

Some obvious properties:
- Color singlets,
- Process independent; i.e. do not depend on the hard scale Q.

J_i not unique (only up to sub-leading soft terms).

A natural scheme: J_i = square root of the space-like QCD formfactor.

Sterman and Tejeda-Yeomans '02

Scheme works in both the massless and the massive cases.

The massive form-factor's exponentiation known through 2 loops

Mitov, Moch '06

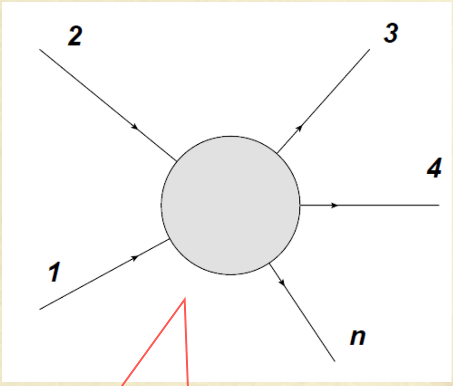
Factorization: the Soft function

$$M_I(Q,m,\epsilon) = J(m,\epsilon) \cdot S_{IJ}(Q,m,\epsilon) \cdot H_J(Q,m)$$

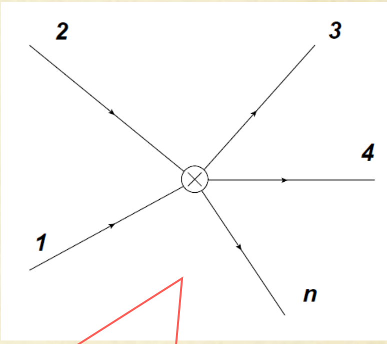
Soft function is the most non-trivial element
(recall: it contains only soft poles).

But we know that the soft limit is reproduced by the eikonal approximation.

➔ Extract S(…) from the eikonalized amplitude:



The LO amplitude M(…)

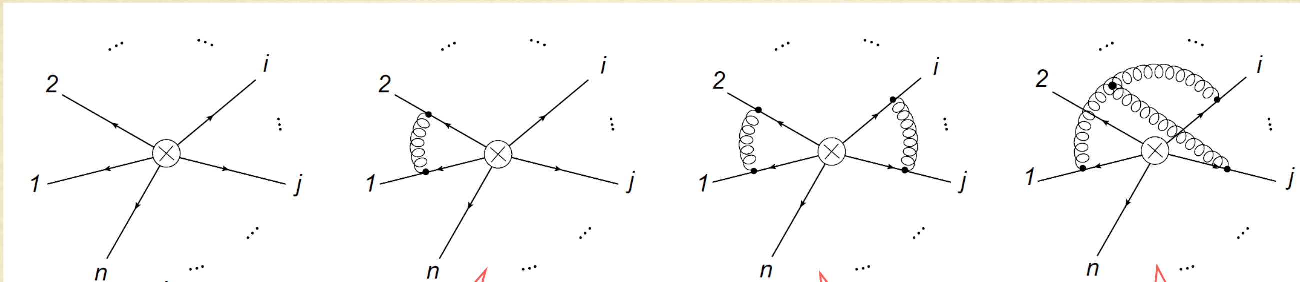


The eikonal version of the amplitude.
(the blob is replaced by an effective n-point vertex)

Factorization: the Soft function

Calculation of the eikonal amplitude:

consider all soft exchanges between the external (hard) partons



The fixed order expansion of the soft function takes the form:

$$\begin{aligned} S^{(1)}_{IJ}(\varepsilon, s_{ij}, m_i) &= \frac{1}{\varepsilon}\Gamma^{(1)}_{IJ}(s_{ij}, m_i) + O(\varepsilon^0)\,, \\ S^{(2)}_{IJ}(\varepsilon, s_{ij}, m_i) &= -\frac{\beta_0}{4\varepsilon^2}\Gamma^{(1)}_{IJ}(s_{ij}, m_i) + \frac{1}{2}\left(S^{(1)}_{IJ}(\varepsilon, s_{ij}, m_i)\right)^2 + \frac{1}{\varepsilon}\Gamma^{(2)}_{IJ}(s_{ij}, m_i) + O(\varepsilon^0)\,. \end{aligned}$$

… as follows from the usual RG equation:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g,\varepsilon)\frac{\partial}{\partial g}\right) S_{IJ}(\varepsilon, s_{ij}, m_i) = -\Gamma_{IK}(\varepsilon, s_{ij}, m_i)\, S_{KJ}(\varepsilon, s_{ij}, m_i)$$

➔ All information about S(…) is contained in the anomal's dimension matrix Γ_{IJ}

the Soft function at 1 loop

Here is the result for the anomalous dim. matrix at one loop

$$\mathbf{\Gamma}_S^{(1)} = \underbrace{\frac{1}{2}\sum_{(i\neq j)=1}^{n} T_i\cdot T_j \ln\left(-\frac{\mu^2}{\sigma_{ij}}\right)}_{\text{The massless case}} + \underbrace{\frac{1}{2}\sum_{(i\neq j)\in\mathcal{N}_m} T_i\cdot T_j \left[\ln\left(1+x_{ij}^2\right) + \frac{2x_{ij}^2}{1-x_{ij}^2}\ln(x_{ij})\right]}_{\text{O(m) corrections in the massive case}}$$

where:
- all masses are taken equal,
- written for space-like kinematics (everything is real).

$$\frac{m^2}{s_{ij}} = -\frac{x_{ij}}{(1-x_{ij})^2} \quad , \quad x_{ij} = \frac{\sqrt{1-\frac{4m^2}{s_{ij}}}-1}{\sqrt{1-\frac{4m^2}{s_{ij}}}+1}$$

$$s_{ij} = (p_i+p_j)^2 \text{ and } \sigma_{ij} = 2p_i\cdot p_j = s_{ij} - m_i^2 - m_j^2$$

The Soft function at 2 loops

The simplest approach is the following. Start with the Ansatz:

$$\underbrace{\mathbf{\Gamma}_S^{(2)} = \frac{1}{2}\sum_{(i\neq j)=1}^{n} T_i \cdot T_j \, \frac{K}{2} \, \ln\left(-\frac{\mu^2}{\sigma_{ij}}\right)}_{\text{Reproduces the massless case}} + \underbrace{\frac{1}{2}\sum_{(i\neq j)\in\mathcal{N}_m} T_i \cdot T_j \, P_{ij}^{(2)}}_{\text{Parametrizes the O(m) corrections to the massless case}} + 3E \text{ terms}$$

Then note: the function $P^{(2)}_{ij}$ depends on (i,j) only through s_{ij}

$\boxed{\rightarrow P^{(2)}_{ij} = P^{(2)}(s_{ij})}$ This single function can be extracted from the known n=2 amplitude: the massive two-loop QCD formfactor.

Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi `04
Gluza, Mitov, Moch, Riemann '09

The Soft function at 2 loops

The complete result for the 2E reads: $P^{(2)} = \frac{K}{2} P^{(1)} + P^{(2),\mathrm{m}}$

$$P^{(2),\mathrm{m}}(x) = \frac{C_A}{(1-x^2)^2}\Bigg\{ -\frac{(1+x^2)^2}{2}\mathrm{Li}_3(x^2) + \left(\frac{(1+x^2)^2}{2}\ln(x) - \frac{1-x^4}{2}\right)\mathrm{Li}_2(x^2)$$
$$+ \frac{x^2(1+x^2)}{3}\ln^3(x) + x^2(1-x^2)\ln^2(x)$$
$$+ \left(-(1-x^4)\ln\left(1-x^2\right) + x^2(1+x^2)\zeta_2\right)\ln(x) + x^2(1-x^2)\zeta_2 + 2x^2\zeta_3\Bigg\},$$

This term breaks the simple relation $\mathbf{\Gamma}^{(2)}_{S_f} = \frac{K}{2}\mathbf{\Gamma}^{(1)}_{S_f}$ from the massless case!

Aybot, Dixon, Sterman ’06

Above result derived by 3 different groups:

Kidonakis ’09
Becher, Neubert ‘09
Czakon, Mitov, Sterman ‘09

Kidonakis derived the massive eikonal formfactor;
Becher, Neubert used old results of Korchemsky, Radushkin

The Soft function at 2 loops

What about the 3E contributions in the massive case?

Until recently there existed no indication if they were non-zero!

In particular, the following squared two-loop amplitudes are insensitive to it:

Czakon, Mitov, Sterman '09

Known numerically

$$\langle M^{(2)}|M^{(0)}\rangle(q\bar{q} \to Q\overline{Q})$$

Czakon '07

Poles reported

$$\langle M^{(2)}|M^{(0)}\rangle(gg \to Q\overline{Q})$$

Czakon, Bärnreuther '09

3E correlators not vanish if at least two legs are massive – direct position-space calculation for Euclidean momenta (numerical results)

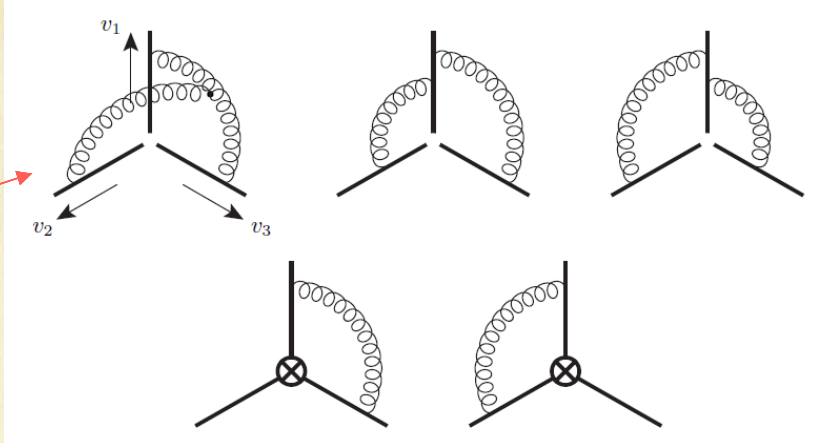
Mitov, Sterman, Sung '09

Exact result computed analytically

Ferroglia, Neubert, Pecjak, Yang '09

The Soft function at 2 loops. Massive case.

The types of contributing diagrams:



The analytical result is very simple:

Ferroglia, Neubert, Pecjak, Yang '09

$$F^{(3g)} \sim \sum_{ijk} \epsilon^{ijk} \ln^2(x_{ij})\, r(x_{ik})$$

where:

$$r(x) = -\frac{1+x^2}{1-x^2}\ln(x)$$

The calculation of the double exchange diagrams is very transparent.
Agrees in both momentum and position spaces

A.M., Sterman, Sung '10

Massive gauge amplitudes: Summary

- The results I presented can be used to predict the poles of any massive 2-loop amplitude with:
  - *n* external colored particles (plus arbitrary number of colorless ones),
  - arbitrary values of the masses (usefull for SUSY).

- Results checked in the 2-loop amplitudes:

$$\langle M^{(2)} | M^{(0)} \rangle (q\bar{q} \to Q\overline{Q})$$
$$\langle M^{(2)} | M^{(0)} \rangle (gg \to Q\overline{Q})$$

- Needed in jet subtractions with massive particles at 2-loops

- Input for NNLL resummation (next slides)

The connection to resummation at hadron colliders

How is the threshold resummation done?

The resummation of soft gluons is driven mostly by kinematics:

Sterman '87
Catani, Trentadue '89

- Only soft emissions possible due to phase space suppression (hence kinematics)
- That's all there is for almost all "standard" processes: Higgs, Drell-Yan, DIS, e^+e^-

Key: the number of hard colored partons < 4

In top pair production (hadron colliders) new feature arises:

Color correlations due to soft exchanges (n>=4)

Non-trivial color algebra in this case.

The top cross-section: NNLL resummation

Factorization of the partonic cross-section close to threshold:

Kidonakis, Sterman '97
Czakon, Mitov, Sterman '09

$$\omega_P\left(N,\hat{\eta},\frac{M^2}{\mu^2},\frac{m^2}{\mu^2},\alpha_s(\mu^2)\right) \;=\; J_1(N,\alpha_s(\mu^2))\ldots J_k(N,M/\mu,m/\mu,\alpha_s(\mu^2))$$
$$\times\ \mathrm{Tr}\left[\mathbf{H}^P\left(\frac{M^2}{\mu^2},\frac{m^2}{\mu^2},\alpha_s(\mu^2)\right)\mathbf{S}^P\left(\frac{N^2\mu^2}{M^2},\frac{M^2}{m^2},\alpha_s(\mu^2)\right)\right]+\mathcal{O}(1/N)$$

N – the usual Mellin dual to the kinematical variable
that defines the threshold kinematics:

$$\sigma(N) \;=\; \int_0^1 dz z^{N-1}\sigma(z)$$

$z = Q^2/s$ ← Drell-Yan

$z = 4m^2/s$ ← t-tbar total X-section

$z = M^2_{t\bar{t}}/s$ ← t-tbar – pair invariant mass

J's – jet functions (different from the ones in amplitudes)

S,H – Soft/Hard functions. Also different.

The top cross-section: NNLL resummation

Specifically, for top-pair production we have:

$$\boxed{\sigma^P(N,m^2,\mu^2) = \sigma^P_{\rm Born}(N)\ \left[J^P_{\rm in}(N,m^2,\mu^2)\right]^2\ \left[J_{\rm incl}(N,m^2,\mu^2)\right]^2\ {\rm Tr}\left[\hat{\mathbf{H}}^P(m^2,\mu^2)\mathbf{S}^P(N,m^2,\mu^2)\right] + \mathcal{O}(1/N)}$$

where:

- $J^P_{\rm in}$ – is the Drell-Yan/Higgs cross-section
- $J_{\rm incl}$ – observable dependent function (i.e. depends on the final state)

$$J_{\rm incl}(N,m^2,\mu^2) = \exp\left\{\frac{1}{2}\int_0^1 dx \frac{x^{N-1}-1}{1-x}\,\Gamma_{\rm incl}\left(\alpha_s\left[4m^2(1-x)^2\right]\right)\right\}$$

$$\Gamma_{\rm incl} = \frac{\alpha_s(\mu^2)}{\pi}\,C_F\left[-1-\ln\left(\frac{m^2}{\mu^2}\right)\right] + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2\left[\frac{K}{2}C_F\left(-1-\ln\left(\frac{m^2}{\mu^2}\right)\right) - \frac{\zeta_3-1}{2}C_F C_A\right]$$

Defines the poles of the massive QCD formfactor in the small-mass limit.

Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi `04
Gluza, Mitov, Moch, Riemann '09
Mitov, Moch '06

The top cross-section: NNLL resummation

Here is the result for the Soft function:

$$
\begin{aligned}
\mathbf{S}\left(\frac{N^2\mu^2}{M^2},\beta_i\cdot\beta_j,\alpha_s(\mu^2)\right)\Bigg|_{\mu=M} &= \overline{\mathcal{P}}\exp\left\{-\int_{M/\bar{N}}^{M}\frac{d\mu'}{\mu'}\mathbf{\Gamma}_S^{\dagger}\left(\beta_i\cdot\beta_j,\alpha_s\left(\mu'^2\right)\right)\right\} \\
&\quad\times\mathbf{S}\left(1,\beta_i\cdot\beta_j,\alpha_s\left(M^2/\bar{N}^2\right)\right) \\
&\quad\times\mathcal{P}\exp\left\{-\int_{M/\bar{N}}^{M}\frac{d\mu'}{\mu'}\mathbf{\Gamma}_S\left(\beta_i\cdot\beta_j,\alpha_s\left(\mu'^2\right)\right)\right\} \\
&= \overline{\mathcal{P}}\exp\left\{\int_0^1 dx\frac{x^{N-1}-1}{1-x}\,\mathbf{\Gamma}_S^{\dagger}\left(\beta_i\cdot\beta_j,\alpha_s\left((1-x)^2M^2\right)\right)\right\} \\
&\quad\times\mathbf{S}\left(1,\beta_i\cdot\beta_j,\alpha_s\left(M^2/N^2\right)\right) \\
&\quad\times\mathcal{P}\exp\left\{\int_0^1 dx\frac{x^{N-1}-1}{1-x}\,\mathbf{\Gamma}_S\left(\beta_i\cdot\beta_j,\alpha_s\left((1-x)^2M^2\right)\right)\right\}
\end{aligned}
$$

Note: the Soft function satisfies RGE with the same anomalous dimension matrix as the Soft function of the underlying amplitude!

Therefore: knowing the singularities of an amplitude, allows resummation of soft logs in observables!

The top cross-section: NNLL resummation

We also need to specify a boundary condition for the soft function:

$$\mathbf{S}\left(1,\beta_i\cdot\beta_j,\alpha_s\left(M^2/N^2\right)\right) = \mathbf{S}^{(0)} + \frac{\alpha_s\left(M^2/N^2\right)}{\pi}\,\mathbf{S}^{(1)}\left(1,\beta_i\cdot\beta_j\right) + \dots$$

For two-loop resummation we need it only at one loop
(since its contribution at two loops is only through the running coupling).

For example, for the total t-tbar cross-section in gg-reaction it reads:

$$\begin{aligned}
\mathbf{S}\left(1,\alpha_s\left(Q^2/N^2\right)\right) &= \mathbf{S}^{(0)}\left[1 + \frac{\alpha_s\left(Q^2/N^2\right)}{\pi}\,C_A\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix} + \dots\right]\\
&= \mathbf{S}^{(0)}\left[1 + C_A\frac{\alpha_s\left(\mu^2\right)}{\pi}\left\{1 + \frac{\alpha_s\left(\mu^2\right)}{\pi}\frac{\beta_0}{4}\ln\left(\frac{N^2\mu^2}{Q^2}\right)\right\}\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix} + \dots\right]
\end{aligned}$$

Can be derived by calculating the one-loop eikonal cross-section.

The top cross-section: NNLL resummation

Combining everything we get the following result for the resummed total t-tbar cross-section:

Hard function. Known exactly at 1 loop.

Czakon, Mitov '08
Hagiwara, Sumino, Yokoya '08

$$\frac{\sigma^P(N,m^2,\mu^2)}{\sigma^P_{\rm Born}(N)} = {\rm Tr}\left[\mathbf{H}^P(m^2,\mu^2)\ \exp\left\{\int_0^1 dx \frac{x^{N-1}-1}{1-x}\right.\right.$$
$$\left.\left.\times\ \left(\int_{\mu_F^2}^{4m^2(1-x)^2}\frac{dq^2}{q^2}\,2\,A_P\left(\alpha_s\left[q^2\right]\right)\mathbf{1}+D^P_{Q\overline{Q}}\left(\alpha_s\left[4m^2(1-x)^2\right]\right)\right)\right\}\right]$$

And the anomalous dimension is:

Jet functions (from Drell-Yan/Higgs)

$$D^P_{Q\overline{Q}} = \frac{\alpha_s(\mu^2)}{\pi}(-C_A)\begin{pmatrix}0&0\\0&1\end{pmatrix}+\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2\left\{D^{(2)}_{\rm P}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\left(-C_A\frac{K}{2}-\frac{\zeta_3-1}{2}C_A^2-C_A\frac{\beta_0}{2}\right)\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$$

Fixed by the small-mass limit of the massive formfactor!

Czakon, Mitov, Sterman '09
Beneke, Falgari, Schwinn '09

Get the cross-section

How we put all this to work?

- Match fixed order and resummed results:

$$\sigma_{RESUM} = \sigma_{NLO} + \sigma_{SUDAKOV} - \sigma_{OVERLAP}$$

Now known at NNLO

- σ_{NLO} is known exactly,
- $\sigma_{SUDAKOV}$: anomalous dimensions and matching coefficients needed.

Known at NLO

i.e. at present one can derive the NLO+NNLL cross-section

Collider phenomenology: Top-pair production

Top-pair cross-section: the threshold expansion

Derive NNLO threshold approximation for the cross-section

- ✓ Use soft-gluon expansion (from resummation)
- ✓ Extract 2-loop Coulombic terms (from, say, e+e- → tT)

Beneke, Czakon, Falgari, Mitov, Schwinn '09

$$\begin{aligned}\sigma_{ij,\mathbf{I}}(\beta,\mu,m) \quad = \quad & \sigma^{(0)}_{ij,\mathbf{I}}\Bigg\{1+\frac{\alpha_s(\mu^2)}{4\pi}\left[\sigma^{(1,0)}_{ij,\mathbf{I}}+\sigma^{(1,1)}_{ij,\mathbf{I}}\ln\left(\frac{\mu^2}{m^2}\right)\right] \\ & +\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2\left[\sigma^{(2,0)}_{ij,\mathbf{I}}+\sigma^{(2,1)}_{ij,\mathbf{I}}\ln\left(\frac{\mu^2}{m^2}\right)+\sigma^{(2,2)}_{ij,\mathbf{I}}\ln^2\left(\frac{\mu^2}{m^2}\right)\right]+\mathcal{O}(\alpha_s^3)\Bigg\}\end{aligned}$$

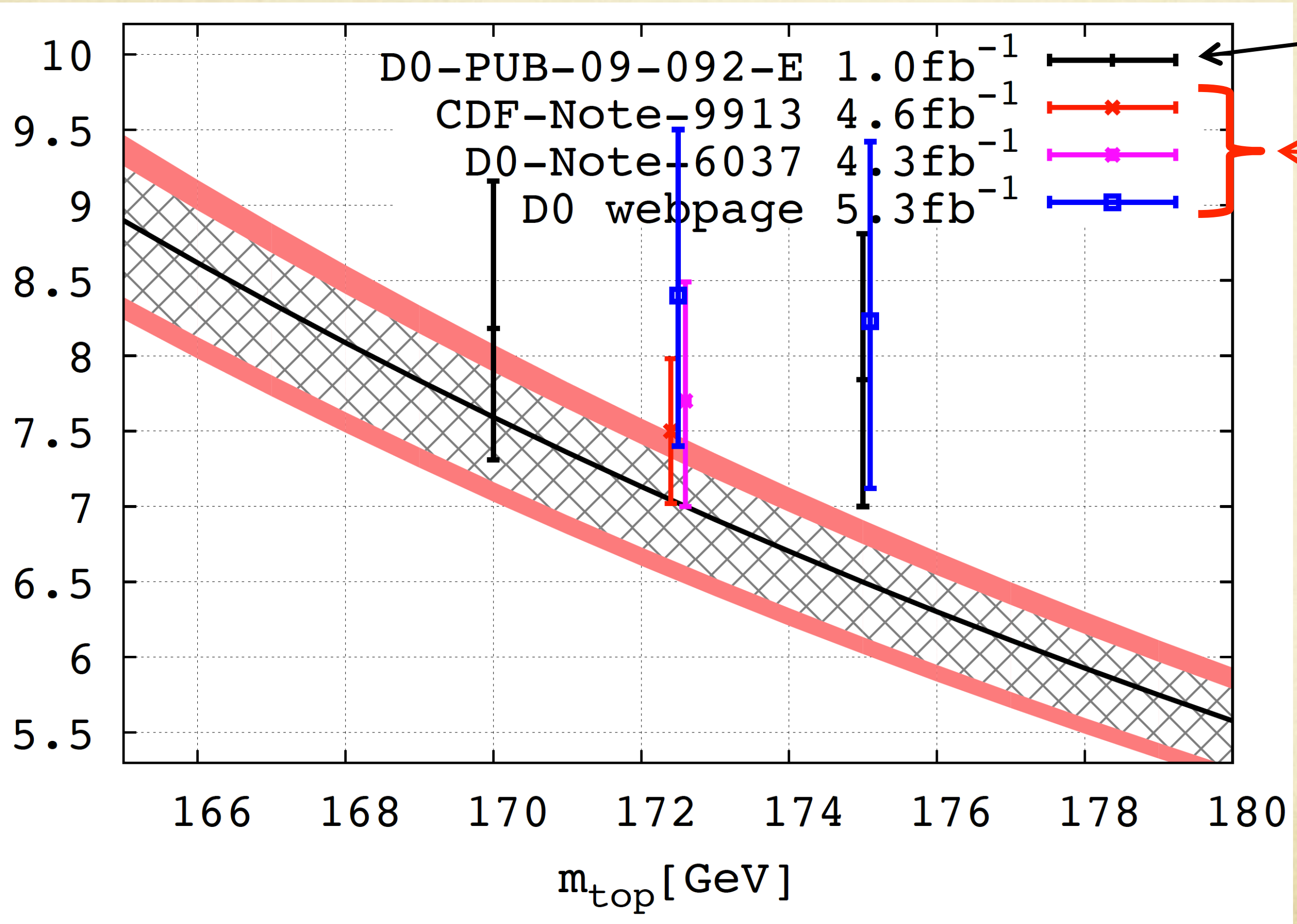
$$\begin{aligned}\sigma^{(2)}_{q\bar{q}} \quad = \quad & \frac{3.60774}{\beta^2}+\frac{1}{\beta}\Big(-140.368\ln^2\beta+32.106\ln\beta+3.95105\Big) \\ & +910.222\ln^4\beta-1315.53\ln^3\beta+592.292\ln^2\beta+528.557\ln\beta+C^{(2)}_{qq}\,,\end{aligned}$$

$$\begin{aligned}\sigma^{(2)}_{gg} \quad = \quad & \frac{68.5471}{\beta^2}+\frac{1}{\beta}\Big(496.3\ln^2\beta+321.137\ln\beta-8.62261\Big) \\ & +4608\ln^4\beta-1894.91\ln^3\beta-912.349\ln^2\beta+2456.74\ln\beta+C^{(2)}_{gg}\,,\end{aligned}$$

Top-pair total X-section: Tevatron numbers

(Preliminary theory)

Best prediction based on NLO+NNLL / NNLO_approx



Published data

Preliminary data

✓ Scale and PDF variation

✓ MSTW2008 (NNLO)

Note O(10%) decrease compared to MRST2006 (NNLO)

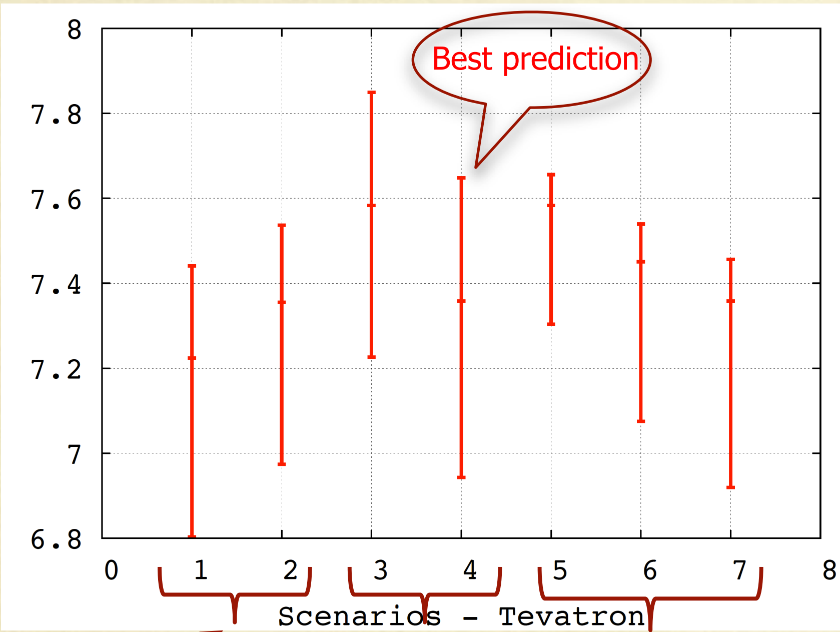
Top-pair total X-section: Tevatron numbers

Being an approximation, how robust are these numbers?

- ✓ Try to understand the physics;
- ✓ Stress-test in all possible ways;
- ✓ Quantify the sensitivities.

Defined on the next slide

Construct a number of NLO+NNLL and NNLO_aprox "scenarios" to analyze:



NNLL Resummation

Approximate NNLO

For m_top=171GeV

Plotted for each scenario are:

- ✓ central values
- ✓ scale uncertainty

Used independent variation of:

- ✓ renormalization scale
- ✓ factorization scale

See Cacciari et al '08

Top-pair total X-section: Tevatron numbers

Two approaches to NNLO_approx (depends on how the unknown const. are treated)

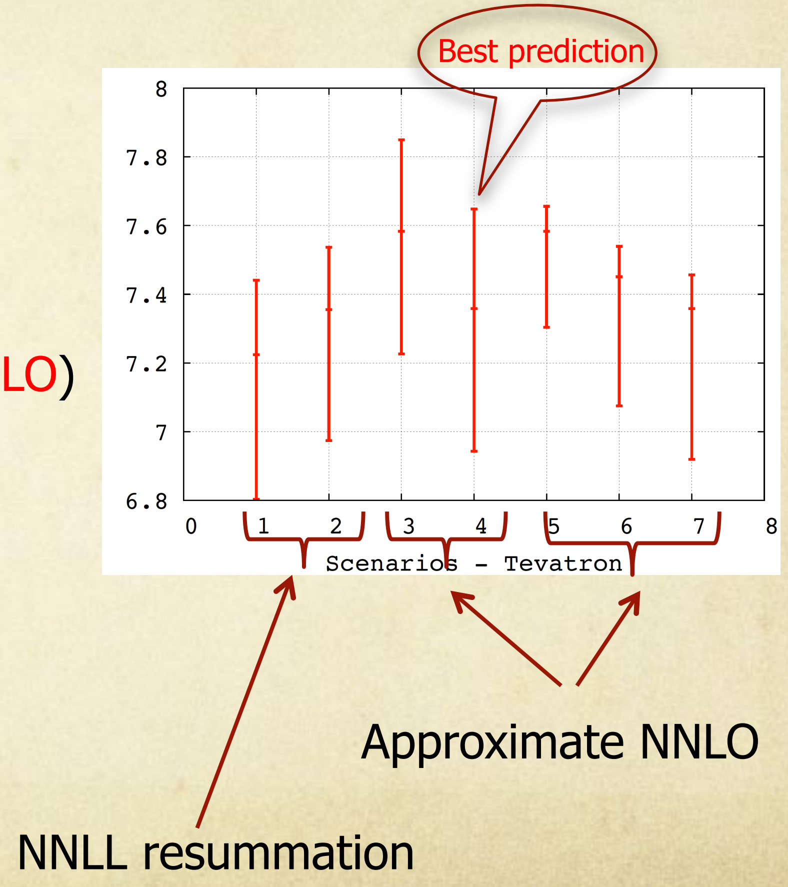
① Unknown constants AND log(mu) terms are omitted:

- ✓ Larger scale variation
- ✓ Consistent approximation
- ✓ Uncertainty = scale variation

② Unknown constants' log(mu) terms INCLUDED:

- ✓ Much smaller scale variation (~ the true NNLO)
- ✓ Uncertainty =
  scale variation AND constant variation
- ✓ Constant varied in a "reasonable range"

Both approaches are mutually consistent



Top-pair total X-section: Tevatron numbers

This picture is fully supported by the known NLO results!

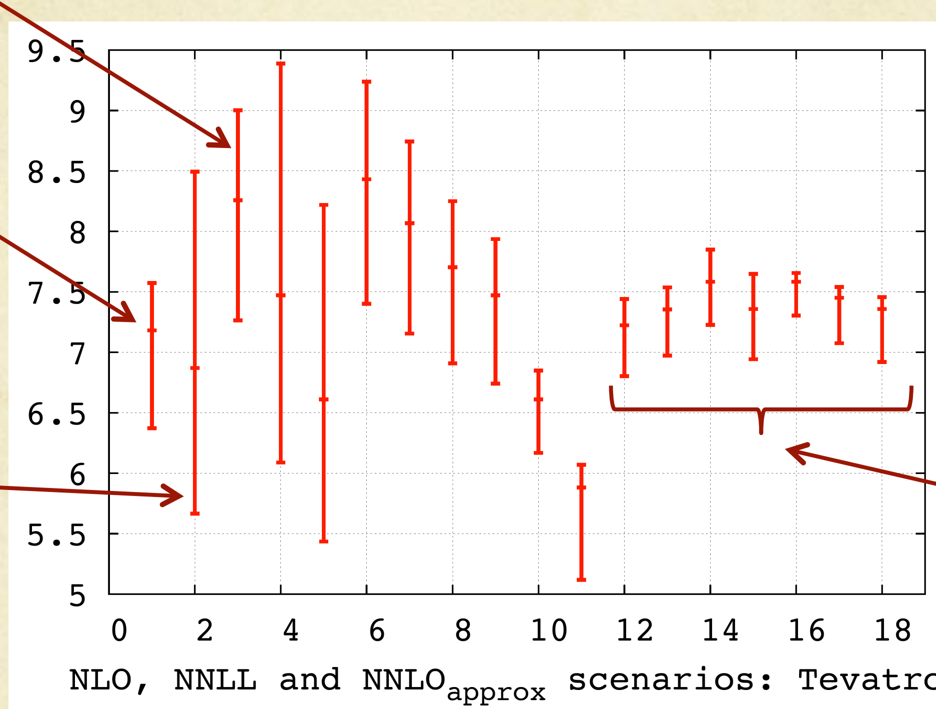
NLO: threshold approximation

For m_top=171GeV

Exact NLO

LO+NLL

NNLL/NNLO_approx approximations



- ✓ NLO/NNLO Pdf sets consistently used
- ✓ Great reduction in sensitivities going from NLO to NNLO

Summary and Conclusions

- New developments in massive gauge theory amplitudes at two loops
  - prediction of the poles of any 2-loop amplitude
- Clarified relation to 2-loop resummation in observables
- Application to top physics at the LHC

Top-pair cross-section

- Analyzed various NNLL and Fixed Order approximations to the NNLO X-section
- Two consistent ways of treating NNLO_approx
- Impressive consistency between all 3 approaches.
- Tevatron results presented; similarities and differences at LHC
- Interesting observations about Coulomb terms; quality of threshold approximation

(Almost) complete understanding of processes with masses at NNLO:

✓ Process-independent info (jets, fragmentation, resummation) (Pdf)

✓ Singular limits